Question 1. Show that if a topological space X has a countable basis $\{B_n\}$, then every basis $\mathcal{C}$ of X contains a countable basis for X.

Answer. Let $\mathcal{B} := \{B_n \,|\, n \in \mathbb{N}\}$ be a basis for the topological space X. Since $\mathcal{C}$ is a basis of X, for every $m \in \mathbb{N}$ there is an element $C \in \mathcal{C}$ such that $C \subset B_m$. Similarly, since $\mathcal{B}$ is a basis of X, there exists an $n \in \mathbb{N}$ such that $B_n \subset C$. Now, for every pair of indices n, m for which it is possible, choose a $C_{n,m}$ such that $B_n \subset C_{n,m} \subset B_m$. Obviously, it is a countable sub collection of $\mathcal{C}$ and we claim that it is also a basis of X. To prove the claim, take an open set U. For any $x \in U$, there exists an m such that $x \in B_m \subset U$. Then, there exists a C such that $x \in C \subset B_m$. Again, there there exists an n such that $x \in B_n \subset C$. Since $B_n \subset C_{n,m} \subset B_m$, we have $x \in B_n \subset C_{n,m} \subset B_m \subset U$. Therefore, the sub-collection of $\mathcal{C}$ is a countable basis of X.

Question 2. Define Hausdorff topological space. Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \,|\, x \in X\}$ is closed in $X \times X$.

Answer. Hausdorff topological space: A topological space X is called Hausdorff if, given any disjoint Points x and y, there are open neighborhoods U of x and V of y that are also disjoint.

At first, assume that X is Hausdorff. We shall prove that $(X \times X) \setminus \Delta$ is open. Let $(p_1, p_2) \in (X \times X) \setminus \Delta$ and hence $p_1 \neq p_2$. Since X is Hausdorff and $p_1, p_2 \in X$, let U_1 and U_2 be disjoint open sets containing p_1 and p_2 respectively. Then $U_1 \times U_2$ is an open set in $X \times X$ containing (p_1, p_2) and such that $(U_1 \times U_2) \cap \Delta = \emptyset$. Therefore, $(p_1, p_2) \in (U_1 \times U_2) \subset (X \times X) \setminus \Delta$. This implies, $(X \times X) \setminus \Delta$ is open, i.e., Δ is closed in $X \times X$.

Finally, assume that Δ is closed, i.e., $(X \times X) \setminus \Delta$ is open. Let $x_1 \neq x_2 \in X$. Then $(x_1, x_2) \in (X \times X) \setminus \Delta$. Since $(X \times X) \setminus \Delta$ is open, there is a open set $U \times V \in X \times X$ such that $(x_1, x_2) \in U \times V \subset (X \times X) \setminus \Delta$. This implies, $x_1 \in U$ and $x_2 \in V$, and U and V are disjoint, which shows that X is Hausdorff.

Question 3. Define connected topological space. Let A and B be proper subsets of connected spaces X and Y respectively. Prove that the complement of $A \times B$ in $X \times Y$ is connected.

Answer: connected topological space: A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected.

Let $x_0 \in X \setminus A$ and $y_0 \in Y \setminus B$, and let C be the connected component of (x_0, y_0) in $X \times Y \setminus A \times B$. We need to show that C is the entire space, and in order to do this it is enough to show that given any other point (x, y) in the space there is a connected subset of $X \times Y \setminus A \times B$ containing it and (x_0, y_0). There are three cases depending upon whether or not $x \in A$ or $y \in B$ (there are three options rather than four because we know that both cannot be true). If $x \notin A$ and $y \notin B$ then the sets $X \times \{y_0\}$ and $\{x\} \times Y$ are connected subsets such that (x_0, y_0) and (x, y_0) lie in the first subset while (x, y_0) and (x, y) lie in the second. Therefore there is a connected subset containing (x, y) and (x_0, y_0). Now suppose that $x \in A$ but $y \notin B$. Then the two points (x_0, y_0) and (x, y) are both contained in the connected subset $X \times \{y\} \cup \{x_0\} \times Y$. Finally, if $x \notin A$ but $y \in B$, then the two points (x_0, y_0) and (x, y) are both contained in the connected subset $X \times \{y_0\} \cup \{x\} \times Y$. Therefore the set $X \times Y \setminus A \times B$ is connected.

Question 4. Define a path connected space. Prove that if U is an open connected subspace of $\mathbb{R}^2$, then U is path connected. Is the result also true for closed subspaces of $\mathbb{R}^2$? Justify your answer.

Answer: path connected topological space: A topological space X is said to be path-connected or arc-wise connected if for any two points $x, y \in X$ there is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

We claim that the set Γ_a of points $x \in U$ such that there is a path $\gamma : [0, 1] \to U$ with $\gamma(0) = a$ and $\gamma(1) = x$ is open and closed in U.

Let x be an element of Γ_a. Then x is connected to a by a path, and there exists an $r > 0$ such that $B(x, r) \subset U$. Then for $y \in B(x, r)$, the map $\tilde{\gamma} : [0, 1] \to U$ defined by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(2t) & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ 2(1-t)x + (2t-1)y & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

is a continuous path joining a to y, so $y \in \Gamma_a$, i.e., $B(x, r) \subset \Gamma_a$. Thus, Γ_a is open.

To prove that it is closed, let $x \in U$ be an accumulation point of Γ_a and $r > 0$ such that $B(x, r) \subset U$. Since x is an accumulation point, there exists $y \neq x$ in $B(x, r) \cap \Gamma_a$. Then, as in the proof that Γ_a is open, one can concatenate a path from a to y and the segment from y to x to get a continuous path in U that connects a to x. It follows that $x \in \Gamma_a$, which means that Γ_a contains its accumulation points, hence is closed in U.

The result is not true for for closed subspaces of $\mathbb{R}^2$. Consider the topologist's sine curve $T = \left\{\left(x, \sin \frac{1}{x}\right) : x \in (0, 1]\right\} \cup \{(0, 0)\}$. The closed topologist's sine curve can be defined by taking the topologist's sine curve and adding its set of limit points, $\{(0, y) \mid y \in [-1, 1]\}\{(0, y) \mid y \in [-1, 1]\}$. This space is closed and, by Example 7, Section 24 of Munkress Topology book., it not path-connected. But by Theorem 23.4 and 23.5 it is connected.

Question 5. If Y is a compact space, then prove that for any space X, the projection map $\pi_1 : X \times Y \to X$ is a closed map.

Answer: We need to show that if $F \subset X \times Y$ is closed then $\pi_1(F)$ is closed in X, and as usual it is enough to show that the complement is open. Suppose that $x \notin \pi_1(F)$. The latter implies that $\{x\} \times Y$ is contained in the open subset $X \times Y \setminus F$, and by the Tube Lemma one can find an open set $V_x \subset X$ such that $x \in V$ and $V_x \times Y \subset X \times Y \setminus F$. But this means that the open set $V_x \subset X$ lies in the complement of $\pi_1(F)$, and since one has a conclusion of this sort for each such x it follows that the complement is open as required.

Question 6. Define normal topological space. Prove that every compact Hausdorff space is normal. Is the converse true? Justify your answer.

Answer: Normal topological space: A topological space X is a normal space if, given any disjoint closed sets E and F, there are open neighborhoods U of E and V of F that are also disjoint.

Every compact Hausdorff space is normal: Theorem 32.3 in Munkress Topology book.

The converse is not true. Consider the real line with usual topology. It is normal but not compact.